More Induction

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Discrete Structures (CS 173 B) Fall 2016

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- Suppose domino k falls. Then domino k+1 falls.

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$$\text{fall}(k) \rightarrow \text{fall}(k+1)$$

- The first domino falls

$$\text{fall}(1) = T$$

Induction

Inductive hypothesis: Suppose domino k falls.

Inductive conclusion: Domino k+1 falls.

Base case: The first domino falls.

Basic structure of induction proof

- Claim: $P(n)$ for all $n = 0, 1, 2, 3 \ldots$

Base: $P(0)$ is true.

Inductive step: $P(k) \rightarrow P(k+1)$ — Weak Induction

Inductive hypothesis ($P(k)$); Inductive conclusion ($P(k+1)$)

or $P(0), P(1), \ldots P(k) \rightarrow P(k+1)$ — Strong Induction

Today's lecture

- More examples of induction proofs
  - Graph coloring
  - Multiple base cases
  - Another strong induction
  - Prime factorization
  - Towns connected by one-way streets

Graph coloring

- Claim: For any positive integer D, if all nodes in a graph have degree $\leq D$, then G can be colored with $D + 1$ colors.

Graph coloring

Claim: For any positive integer D, if all nodes in a graph have degree $\leq D$, then G can be colored with $D + 1$ colors.

Base case: A graph that has 1 node has maximum degree less than or equal to 0 and can be colored with 1 color.

Induction: Suppose that any graph with at most $k - 1$ nodes with maximum degree $\leq D$ can be colored with $D + 1$ colors. We need to show that any graph of k nodes with maximum degree $\leq D$ can also be colored with $D + 1$ colors.

Let G be a graph with k nodes and degree $\leq D$. Remove one node v and its edges from the graph to form a subgraph G' with $k - 1$ nodes and at most degree D (since removing edges can't increase the degree). By the inductive hypothesis G' can be colored with $D + 1$ colors. Node v is connected to at most D other nodes, so it can be assigned the remaining $(D + 1)$th color.

QED

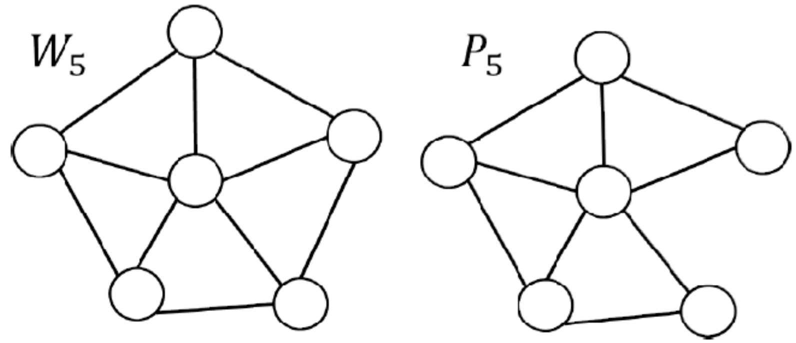

Figure 1: Illustration of pizza graph P_n. A pizza graph is a wheel graph with one edge removed from the cycle.

Define a pizza graph P_n as a wheel graph W_n with one of the edges on the cycle removed, so-called because it looks like a round pizza with one slice missing (pizza graph is not a standard term in graph theory). Using strong induction, prove that graph P_n, which contains $n+1$ nodes in total with $n \geq 3$, contains exactly $2n-1$ edges.

Note: Your proof should not make use of any properties of W_n.

Postage example (with strong induction)

Claim: Every amount of postage that is at least 12 cents can be made from 4- and 5-cent stamps.

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I.e., show that $n = 4a + 5b$ for some natural numbers a and b and any integers $n \geq 12$.

Base cases:

$$n = 12: a = 3, b = 0 \Rightarrow 4 * 3 + 5 * 0 = 12$$
$$n = 13: a = 2, b = 1 \Rightarrow 4 * 2 + 5 * 1 = 13$$
$$n = 14: a = 1, b = 2 \Rightarrow 4 * 1 + 5 * 2 = 14$$
$$n = 15: a = 0, b = 3 \Rightarrow 4 * 0 + 5 * 3 = 15$$

Induction: Suppose that there exist natural integers a and b such that $n = 4a + 5b$ for each integer $n \in \{12, 13, \ldots k - 1\}$, where $k \geq 16$. Then, we have to show that $k = 4i + 5j$ for some natural i and j. Since $k \geq 16$, $k - 4 \geq 12$, so $k - 4 = 4a + 5b$ for some natural a, b. So $k = 4(a + 1) + 5b$.

QED

In the country of Discretia, the secretary of treasury Uomi Lots has gone a bit mad and decided to print only \$3 and \$5 bills. Before he is lynched, Uomi argues that two people with enough of both kinds of bills can transfer any natural integer amount of money. For example, if Mary wants to give Sue \$2, she can give Sue one \$5 bill and take one \$3 bill.

Using a strong inductive hypothesis and a sufficient set of base cases, prove Uomi's assertion:

Claim: For any $n \in \mathbb{N}$, there exist integers a and b, such that $n = 3a + 5b$.

How many base cases?

Strengthening of induction hypothesis

Prove that sum of the first n odd numbers is a perfect square.

Strengthening of induction hypothesis

Prove that sum of the first n odd numbers is n^2.

Avoid induction step in opposite direction

In every arrangement of n lines in the plane the areas can be colored using two colors such that no two neighboring areas are colored the same.

Nim

Nim: Two piles, two piles of matches. Each player takes turns removing any number of matches from either pile. Player that takes last match wins.

Claim: If the two piles contain the same number of matches at the start of the game, then the second player can always win.

Prime factorization

Claim: Every positive integer can be written as the product of primes.

Tips for induction

- Induction always involves proving a claim for a set of integers (e.g., number of nodes in a graph)

- Sketch out a few simple cases to help determine the base case and strategy for induction
  - How many base cases are needed?
  - How does the next case follow from the base cases?

- Carefully write the full inductive hypothesis and what you need to show

- Make sure that your induction step uses the inductive hypothesis to reach the conclusion

Next class

- Recursive definitions